

THE INVERSE SURFACES OF TANGENT DEVELOPABLES WITH RESPECT TO $S_c(r)$

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Abstract. In this paper, we define the inverse surface of a tangent developable surface with respect to the sphere $S_c(r)$ with the center $c \in \mathbb{E}^3$ and the radius r in 3-dimensional Euclidean space \mathbb{E}^3 . We obtain the curvatures, the Christoffel symbols and the shape operator of this inverse surface by the help of these of the tangent developable surface. Moreover, we give some necessary and sufficient conditions regarding the inverse surface being flat and minimal.

Keywords. Inversion, Inverse surface, Developable surface, Fundamental forms, Christoffel symbols.

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1 Introduction

The last ten years, the developable ruled surfaces are studied by many mathematicians. Developable surfaces are a type of important and fundamental surfaces universally used in industry design. Different methods have been presented for the design of developable surfaces. The use of developable surfaces in ship design is of engineering importance because they can be easily manufactured without stretching or tearing, or without the use of heat treatment. In some cases, a ship hull can be entirely designed with the use of developable surfaces. See, [4, 8, 9, 10, 11].

On the other hand, a conformal map is a function which preserves the angles. The conformal mapping is an important technique used in complex analysis and has many applications in different physical situations.

An inversion with respect to the sphere $S_c(r)$ with the center $c \in \mathbb{E}^3$ and the radius r given by

$$c + \frac{r^2}{\|p - c\|^2} (p - c),$$

$p \in \mathbb{E}^3$, is a conformal mapping and also is differentiable. In \mathbb{E}^3 , the inversion is a transformation defining between open subsets of \mathbb{E}^3 .

In this paper, we firstly tell inversions and inversions of surfaces in \mathbb{E}^3 . Next, we give the fundamental forms, the curvatures(Gauss and mean), the shape operator and the Christoffel symbols of the tangent developable. Finally, using by these properties, we obtain these of the inverse surface of the tangent developable.

2 Basic notions of inverse surfaces

Let $c \in \mathbb{E}^3$ and $r \in \mathbb{R}^+$. We denote that $(\mathbb{E}^3)^* = \mathbb{E}^3 - \{c\}$. Then, an inversion of \mathbb{E}^3 with the center $c \in \mathbb{E}^3$ and the radius r is the map

$$\Phi[c, r] : (\mathbb{E}^3)^* \longrightarrow (\mathbb{E}^3)^*$$

given by

$$\Phi[c, r](p) = c + \frac{r^2}{\|p - c\|^2} (p - c). \quad (2.1)$$

Definition 2.1. ([7]) Let $\Phi[c, r]$ be an inversion with the center c and the radius r . Then, the tangent map of Φ at $p \in (\mathbb{E}^3)^*$ is the map

$$\Phi_{*p} : T_p((\mathbb{E}^3)^*) \longrightarrow T_{\Phi(p)}((\mathbb{E}^3)^*)$$

given by

$$\Phi_{*p}(v_p) = \frac{r^2 v_p}{\|p - c\|^2} - \frac{2r^2 \langle p - c, v_p \rangle}{\|p - c\|^4} (p - c),$$

where $v_p \in T_p((\mathbb{E}^3)^*)$.

Now, let us assume that $\mathfrak{X} : U \subset \mathbb{E}^2 \longrightarrow (\mathbb{E}^3)^*$ is the parametrization of a surface. The inverse surface of \mathfrak{X} with respect to $\Phi[c, r]$ is the surface given by

$$\mathfrak{Y} = \Phi[c, r] \circ \mathfrak{X}, \quad (2.2)$$

Throughout this paper, we assume that Φ is an inversion of \mathbb{E}^3 with the center c and the radius r , \mathfrak{X} is a patch in $(\mathbb{E}^3)^*$ and \mathfrak{Y} is inverse patch of \mathfrak{X} with respect to Φ .

Let $I_{\mathfrak{X}}$, $II_{\mathfrak{X}}$ and $K_{\mathfrak{X}}$, $H_{\mathfrak{X}}$ be the first and second fundamental forms and the curvatures (Gauss and mean) of \mathfrak{X} , and let $I_{\mathfrak{Y}}$, $II_{\mathfrak{Y}}$ and $K_{\mathfrak{Y}}$, $H_{\mathfrak{Y}}$ be these of \mathfrak{Y} , respectively. From [1], we have

$$I_{\mathfrak{Y}} \circ \Phi_* = \lambda^2 I_{\mathfrak{X}}, \quad (2.3)$$

$$II_{\mathfrak{Y}} \circ \Phi_* = -\lambda II_{\mathfrak{X}} - 2\delta I_{\mathfrak{X}}, \quad (2.4)$$

$$K_{\mathfrak{Y}} = \frac{1}{\lambda^2} K_{\mathfrak{X}} + \frac{4}{r^2} \lambda^{-1} \eta H_{\mathfrak{X}} + \frac{4}{r^4} \eta^2, \quad (2.5)$$

$$H_{\mathfrak{Y}} = -\frac{1}{\lambda}H_{\mathfrak{X}} - \frac{2\eta}{r^2}, \quad (2.6)$$

where $\lambda = \frac{r^2}{\|\mathfrak{X}-c\|^2}$, $\delta = \frac{2r^2\langle U_{\mathfrak{X}}, (\mathfrak{X}-c) \rangle}{\|\mathfrak{X}-c\|^4}$ and $\eta = \langle U_{\mathfrak{X}}, (\mathfrak{X}-c) \rangle$.

3 The tangent developable surface

Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a curve with arc-length s and $\{T, N, B\}$ be Frenet frame along γ . Denote by κ and τ the curvature and the torsion of the curve γ , respectively. Then we have Frenet formulas

$$\begin{aligned} T'(s) &= \kappa(s) N(s), \\ N'(s) &= -\kappa(s) T(s) + \tau(s) B(s), \\ B'(s) &= -\tau(s) N(s). \end{aligned}$$

The tangent developable of γ is a ruled surface parametrized by

$$\mathfrak{M}(s, u) = \gamma(s) + uT(s), \quad (3.1)$$

where T is unit tangent vector field of γ . As it is known, the coefficients of the first and second fundamental forms of the surface $\mathfrak{M}(s, u)$ have following

$$E_{\mathfrak{M}} = 1 + (u\kappa)^2, \quad F_{\mathfrak{M}} = G_{\mathfrak{M}} = 1, \quad (3.2)$$

and

$$e_{\mathfrak{M}} = -\operatorname{sgn}(u\kappa)(u\kappa\tau), \quad f_{\mathfrak{M}} = g_{\mathfrak{M}} = 0. \quad (3.3)$$

The normal vector field of the surface $\mathfrak{M}(s, u)$ is given by

$$U_{\mathfrak{M}}(s, u) = -\operatorname{sgn}(u\kappa) B(s). \quad (3.4)$$

Next the curvatures (mean and Gaussian) and the matrix of shape operator of this surface are respectively as follows

$$H_{\mathfrak{M}} = \frac{-\operatorname{sgn}(u\kappa)\tau}{2u\kappa}, \quad \text{and} \quad K_{\mathfrak{M}} = 0 \quad (3.5)$$

and

$$S_{\mathfrak{M}} = \left(\frac{-\operatorname{sgn}(u\kappa)\tau}{2u\kappa} \right) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \quad (3.6)$$

Finally, the Christoffel symbols of the surface $\mathfrak{M}(s, u)$ are given by

$$\begin{aligned} (\Gamma_{11}^1)_{\mathfrak{M}} &= \frac{u\kappa_s + \kappa}{u\kappa}, \\ (\Gamma_{11}^2)_{\mathfrak{M}} &= \frac{-\kappa(1 + (u\kappa)^2) - u\kappa_s}{u\kappa}, \\ (\Gamma_{12}^1)_{\mathfrak{M}} &= -(\Gamma_{12}^2)_{\mathfrak{M}} = \frac{1}{u}, \\ (\Gamma_{22}^1)_{\mathfrak{M}} &= (\Gamma_{22}^2)_{\mathfrak{M}} = 0. \end{aligned} \quad (3.7)$$

4 The inverse surface of the tangent developable

We show that \mathfrak{N} is the inverse surface of the tangent developable surface \mathfrak{M} with respect to the inversion Φ . Thus the inverse surface \mathfrak{N} has following parametrization

$$\mathfrak{N} = c + \frac{r^2}{\|\mathfrak{M} - c\|^2} (\mathfrak{M} - c). \quad (4.1)$$

Hence, if we take into account the equalities (2.3) and (2.4), then the coefficients of the first and second fundamental forms of the inverse surface \mathfrak{N} by the help of these of the surface \mathfrak{M} are given by

$$E_{\mathfrak{N}} = \lambda^2 (1 + (u\kappa)^2), \quad F_{\mathfrak{N}} = G_{\mathfrak{N}} = \lambda^2, \quad (4.2)$$

$$l_{\mathfrak{N}} = \operatorname{sgn}(u\kappa) \lambda u \kappa \tau - 2\delta (1 + (u\kappa)^2), \quad m_{\mathfrak{N}} = n_{\mathfrak{N}} = -2\delta, \quad (4.3)$$

where $E_{\mathfrak{N}}$, $F_{\mathfrak{N}}$, $G_{\mathfrak{N}}$ and $l_{\mathfrak{N}}$, $m_{\mathfrak{N}}$, $n_{\mathfrak{N}}$ are the coefficients of the first and second fundamental forms of the inverse surface \mathfrak{N} , respectively.

Moreover, the Gauss and mean curvatures of the inverse surface \mathfrak{N} by the help of these of the surface \mathfrak{M} are respectively, using by (2.5) and (2.6),

$$K_{\mathfrak{N}} = \frac{4}{r^2} \eta \left(-\operatorname{sgn}(u\kappa) \frac{\tau}{2\lambda u \kappa} + \frac{\eta}{r^2} \right), \quad (4.4)$$

$$H_{\mathfrak{N}} = \operatorname{sgn}(u\kappa) \frac{\tau}{2\lambda u \kappa} - \frac{2\eta}{r^2}, \quad (4.5)$$

where $K_{\mathfrak{N}}$ and $H_{\mathfrak{N}}$ are the Gauss and the mean curvatures of the inverse surface \mathfrak{N} , respectively.

Theorem 4.1. Let \mathfrak{N} be the inverse surface of the tangent developable surface \mathfrak{M} with respect to the inversion Φ . Denote by $S_{\mathfrak{N}}$ the matrix of the shape operator of the inverse surface \mathfrak{N} , then $S_{\mathfrak{N}}$ is given by the help of that of \mathfrak{M} as follows

$$S_{\mathfrak{N}} = \begin{bmatrix} \operatorname{sgn}(u\kappa) \frac{\tau}{\lambda u \kappa} - \frac{2\eta}{r^2} & \operatorname{sgn}(v\kappa) \frac{\tau}{\lambda v \kappa} \\ 0 & -\frac{2\eta}{r^2} \end{bmatrix}. \quad (4.6)$$

Proof. Let $S_{\mathfrak{M}}$ be the matrix of the shape operator of surface \mathfrak{M} . By using the equalities (2.3) and (2.4), we can write

$$S_{\mathfrak{N}} \circ \Phi_* = -\lambda^{-1} S_{\mathfrak{M}} - \frac{2}{r^2} \eta I_2, \quad (4.7)$$

where I_2 is identity, $\lambda = \frac{r^2}{\|\mathfrak{M} - c\|^2}$ and $\eta = \langle U_{\mathfrak{M}}, (\mathfrak{M} - c) \rangle$. Hence from (3.6) and (4.7), we obtain that the equality (4.6) is satisfied.

Theorem 4.2. Let $(\Gamma_{jk}^i)_{\mathfrak{N}}$ be the Christoffel symbols of the inverse surface \mathfrak{N} . The Christoffel symbols of the inverse surface \mathfrak{N} by the help of these of the surface \mathfrak{M} are given by

$$\begin{aligned}
(\Gamma_{11}^1)_{\mathfrak{N}} &= \frac{u\kappa_s + \kappa}{u\kappa} + \frac{\left((u\kappa)^2 - 1\right) \frac{\partial \lambda^2}{\partial s} + \left((u\kappa)^2 + 1\right) \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 (u\kappa)^2}, \\
(\Gamma_{11}^2)_{\mathfrak{N}} &= \frac{-\kappa \left(1 + (u\kappa)^2\right) - u\kappa_s}{u\kappa} + \frac{\left((u\kappa)^2 + 1\right) \frac{\partial \lambda^2}{\partial s} + \left((u\kappa)^2 + 1\right)^2 \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 (u\kappa)^2}, \\
(\Gamma_{12}^1)_{\mathfrak{N}} &= \frac{1}{u} + \frac{\left((u\kappa)^2 + 1\right) \frac{\partial \lambda^2}{\partial u} - \frac{\partial \lambda^2}{\partial s}}{2\lambda^2 (u\kappa)^2}, \\
(\Gamma_{12}^2)_{\mathfrak{N}} &= -\frac{1}{u} + \frac{\left((u\kappa)^2 + 1\right) \left(\frac{\partial \lambda^2}{\partial s} - \frac{\partial \lambda^2}{\partial u}\right)}{2\lambda^2 (u\kappa)^2}, \\
(\Gamma_{22}^1)_{\mathfrak{N}} &= \frac{\left(\frac{\partial \lambda^2}{\partial u} - \frac{\partial \lambda^2}{\partial s}\right)}{2\lambda^2 (u\kappa)^2}, \\
(\Gamma_{22}^2)_{\mathfrak{N}} &= \frac{\left((u\kappa)^2 - 1\right) \frac{\partial \lambda^2}{\partial u} + \frac{\partial \lambda^2}{\partial s}}{2\lambda^2 (u\kappa)^2}.
\end{aligned}$$

Proof. Considering the equality (2.3), for $i, j, k = 1$, we can write

$$(\Gamma_{11}^1)_{\mathfrak{N}} = (\Gamma_{11}^1)_{\mathfrak{M}} + \frac{[E_{\mathfrak{M}}G_{\mathfrak{M}} - 2F_{\mathfrak{M}}^2] \frac{\partial \lambda^2}{\partial s} + F_{\mathfrak{M}}E_{\mathfrak{M}} \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 [E_{\mathfrak{M}}G_{\mathfrak{M}} - F_{\mathfrak{M}}^2]},$$

where $\lambda = \frac{r^2}{\|\mathfrak{M}-c\|^2}$ and $(\Gamma_{11}^1)_{\mathfrak{M}}$ is the Christoffel symbol of the tangent developable surface. Thus, from the equalities (3.2) and (3.7), we obtain

$$(\Gamma_{11}^1)_{\mathfrak{N}} = \frac{u\kappa_s + \kappa}{u\kappa} + \frac{\left((u\kappa)^2 - 1\right) \frac{\partial \lambda^2}{\partial s} + \left((u\kappa)^2 + 1\right) \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 (u\kappa)^2}$$

Others are found in similar way.

Theorem 4.3 Let \mathfrak{N} be the inverse surface of the tangent developable surface \mathfrak{M} with respect to the inversion Φ . Then the inverse surface \mathfrak{N} is a flat surface if and only if either the normal lines to the surface \mathfrak{M} or the tangent planes of the surface \mathfrak{M} pass through the center of inversion.

Proof. Let us assume that the inverse surface \mathfrak{N} is flat, then from (4.4), we can write

$$\frac{4}{r^2}\eta \left(-\operatorname{sgn}(u\kappa) \frac{\tau}{2\lambda u\kappa} + \frac{\eta}{r^2} \right) = 0, \quad (4.9)$$

where either

$$\eta = \langle U_{\mathfrak{M}}, (\mathfrak{M} - c) \rangle = 0, \quad (4.10)$$

or

$$\operatorname{sgn}(u\kappa) \frac{\tau}{2\lambda u\kappa} = \frac{\eta}{r^2}. \quad (4.11)$$

If the equality (4.10) is satisfied, then the tangent planes of the surface \mathfrak{M} pass through the center of inversion. If the equality (4.11) holds, then it follows

$$U_{\mathfrak{M}} = \operatorname{sgn}(u\kappa) \frac{\tau}{2u\kappa} (\mathfrak{M} - c).$$

Namely, the normal lines to the surface \mathfrak{M} pass through the center of inversion.

The proof of sufficient condition is obvious.

Theorem 4.4 Let \mathfrak{N} be the inverse surface of the tangent developable surface \mathfrak{M} with respect to $S_c(r)$. The inverse surface \mathfrak{N} is minimal if and only if the normal lines to the surface \mathfrak{M} pass through the center of inversion

Proof. The proof is same with that of Theorem 4.1.

Applications.

Fig 1.

Fig 2.

Fig 1: The helicoid given by $(u \cos v, u \sin v, 2v)$.

Fig 2: The inverse surface of the helicoid with respect to unit sphere given by

$$\left(\frac{u}{u^2 + 4v^2} \cos v, \frac{u}{u^2 + 4v^2} \sin v, \frac{2v}{u^2 + 4v^2} \right).$$

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